What can the Taylor series do for us?

- If $f: D \to \mathbb{C}$ is a complex-valued fcn on our domain D, and if $\forall z_0 \in D$, $\exists r > 0$ s.t. $f(z) = \sum_{k \geq 0} c_k (z - z_0)^k$ for $z \in B(z_0, r)$ for some constants c_k, then f is holomorphic!.

- We say a zero z_0 of a holomorphic function g has order $m \in \mathbb{N}$ if:

$$g(z_0) = 0 = g'(z_0) = g''(z_0) = \dots = g^{(m-1)}(z_0)$$

and $g^{(m)}(z_0) \neq 0$.

[z_0 is a zero of $g \Leftrightarrow g(z_0) = 0$.].

<u>Thm.</u> Let $g: D \to \mathbb{C}$ be a holomorphic function on a domain D that is not identically zero. Then the zeros of g are isolated.

[We say a zero z_0 of g is isolated if $\exists \varepsilon > 0$ s.t. $g(z) \neq 0$ for $z \in B(z_0, \varepsilon) \setminus \{z_0\}$
$= B^*(z_0, \varepsilon)$.]

z_0 is the only place where $g(z)=0$ in $\boxed{\text{here}}$.

<u>Proof</u>: If z_0 is a zero of the function g as above, then the Taylor series is not identically zero (otherwise the function would be.).

Thus, for some $r>0$, $B(z_0,r)\subseteq D$, and

$$g(z) = \sum_{k\geq 0} c_k (z-z_0)^k = c_m (z-z_0)^m + c_{m+1}(z-z_0)^{m+1} + \cdots$$

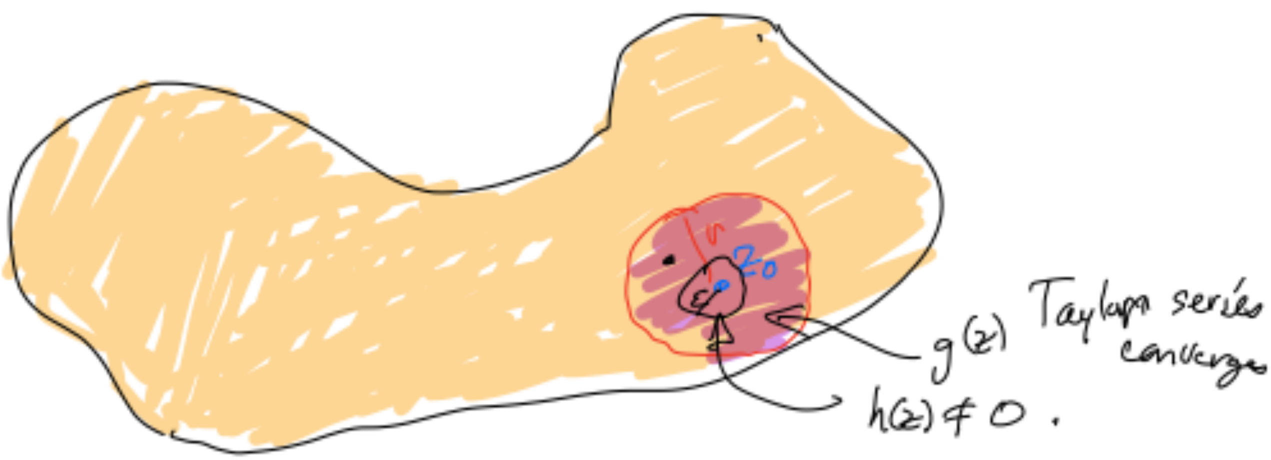
where $c_m \neq 0$.

The first nonzero coefficient.

$$\Rightarrow g(z) = (z-z_0)^m \underbrace{\left[c_m + c_{m+1}(z-z_0) + \cdots \right]}_{\text{convergent Taylor series to } h(z) \text{ — holomorphic near } z=z_0.}$$

Note: $h(z_0) = c_m \neq 0$.

Since h is holomorphic, it is continuous, and so $\exists\, \varepsilon > 0$ s.t. $\underline{h(z) \neq 0}$ $\forall z \in B(z_0, \varepsilon)$.



Recap: for $z \in B(z_0,\varepsilon)$,

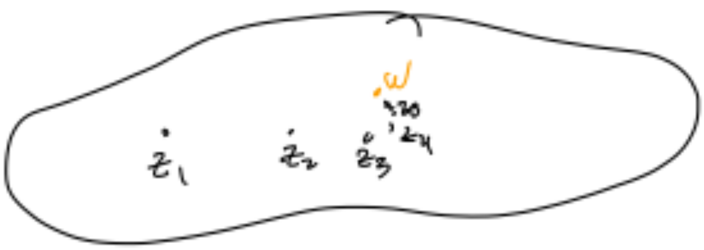
$$g(z) = \underbrace{(z-z_0)^m}_{\text{zero only at } z_0} \underbrace{(h(z))}_{\neq 0}$$

Nonzero for all $z \in B^*(z_0,\varepsilon)$.

$\therefore$ z_0 is an isolated zero. ∎

<u>Corollary (Identity Theorem for Holomorphic fcns.)</u>

Let $f: D \to \mathbb{C}$ be a holomorphic fcn on a domain D. Suppose that there exists a sequence $(z_1, z_2, \ldots)$ of zeros of f converging to a point $w \in D$. Then $f(z) = 0$ $\forall z \in D$. (f is identically zero in D).



<u>Proof:</u> Let $w = \lim_{n\to\infty} z_n$. Then

$$f\left(\lim_{n\to\infty}(z_n)\right) = f(w) = \lim_{n\to\infty} f(z_n) \quad \text{because } f \text{ is continuous}$$

$$= \lim_{n\to\infty} 0 = 0.$$

$\therefore$ w is not an isolated zero. By the last theorem, f is identically zero on D.

(So no holomorphic fn looks like this on $\mathbb{R}$: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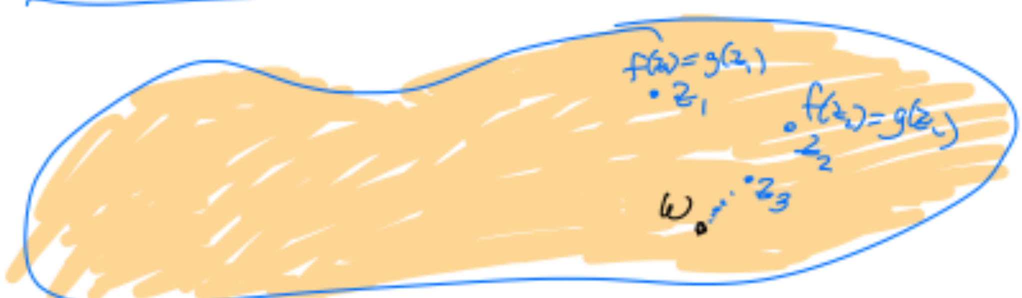
Cor (The Uniqueness Theorem for holomorphic fcns).

Suppose that f, g are holom. fcns on domain $D \subseteq \mathbb{C}$. Suppose $\exists$ convergent sequence of points (z_n) in D s.t. $\lim_{n\to\infty} z_n = \omega \in D$ and $f(z_n) = g(z_n)$ for all $n \in \mathbb{N}$.

Then $f(z) = g(z)$ $\forall z \in D$.



Proof: Consider $f-g : D \to \mathbb{C}$.
$(f-g)(z_n) = 0$ $\forall n$. Apply the

last corollary: $f(z) - g(z) = 0$

$\forall z \in D \ldots$

$\Rightarrow f(z) = g(z) \;\forall z \in D. \;\blacksquare$

I forgot to mention:

Since $f(z) = f(z_0) + f'(z_0)(z - z_0)$

$+ \frac{f''(z_0)}{2!}(z - z_0)^2 + \cdots$,

the Taylor series of a fcn is unique

— coefficients are determined by the derivatives.